This question paper contains 3 printed pages.

Your Roll No. ....

Sl. No. of Ques. Paper: **5715** **H**

Unique Paper Code : **235505**

Name of Paper : **Linear Programming and Theory of Games (MAHT-504)**

Name of Course : **B.Sc. (Hons.) Mathematics**

Semester : **V**

Duration : **3 hours**

Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any two parts from each question.
All questions carry equal marks.

1. (a) If the L.P.P. is

Maximize $z = cx$

subject to

$$Ax = b$$
$$x \geq 0$$

$(x_B, 0)$ is a basic feasible solution corresponding to basis B having an a_j with $z_j - c_j < 0$ and all corresponding $y_{ij} \leq 0$, then show that the L.P.P. has an unbounded solution.

(b) Consider the following linear programming problem:

Maximize
$$z = -3x_1 - 2x_2$$
subject to
$$-x_1 + x_2 \leq 1$$
$$5x_1 + 3x_2 \leq 15$$
$$x_1 \geq 0$$
$$x_2 \geq \tfrac{3}{2}.$$

Solve the problem graphically.

(c) Consider the following system

$$x_1 + x_2 + x_3 \leq 2$$
$$-x_1 + 2x_2 + 2x_3 \leq 3$$
$$x_1,\ x_2,\ x_3 \geq 0$$

The point $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is feasible. Verify whether it is basic. If not, then reduce it to a basic feasible solution.

2. (a) Solve the following problem by simplex method:

Maximize
$$z = x_1 - 2x_2 + x_3$$
subject to

P. T. O.

$$x_1 + 2x_2 + x_3 \leq 12$$
$$2x_1 + x_2 - x_3 \leq 6$$
$$-x_1 + 3x_2 \leq 9$$
$$x_1,\ x_2,\ x_3 \geq 0.$$

(b) Use Big M method to solve the following linear programming problem:

Maximize
$$z = 6x_1 + 4x_2$$
subject to the constraints
$$2x_1 + 3x_2 \leq 30$$
$$3x_1 + 2x_2 \leq 24$$
$$x_1 + x_2 \geq 3$$
$$x_1,\ x_2 \geq 0.$$

(c) Show that there does not exist any feasible solution to the linear programming problem:

Maximize
$$z = 2x_1 + 3x_2 + 5x_3$$
subject to the constraints
$$3x_1 + 10x_2 + 5x_3 \leq 15$$
$$33x_1 - 10x_2 + 9x_3 \leq 33$$
$$x_1 + 2x_2 + x_3 \geq 4$$
$$x_1,\ x_2,\ x_3 \geq 0.$$

3. (a) If $(x_B, 0)$ is a primal optimal solution, then show that there exists a dual optimal solution w, such that $c_B x_B = b^T w$.

(b) Maximize
$$z = x_1 - x_2 + 3x_3 + 2x_4$$
subject to the constraints
$$x_1 + x_2 \geq -1$$
$$x_1 - 3x_2 - x_3 \leq 7$$
$$x_1 + x_3 - 3x_4 = -2$$
$$x_1,\ x_4 \geq 0$$
$$x_2,\ x_3 \text{ unrestricted}$$

(c) Use duality to solve the following L.P.P.:

Maximize
$$z = 2x_1 + x_2$$
subject to the constraints
$$x_1 + 2x_2 \leq 10$$
$$x_1 + x_2 \leq 6$$
$$x_1 - x_2 \leq 2$$
$$x_1 - 2x_2 \leq 1$$
$$x_1,\ x_2 \geq 0$$

4. (a) Solve the following assignment problem:

	I	II	III	IV	V
A	3	5	10	15	8
B	4	7	15	18	8
C	8	12	20	20	12
D	5	5	8	10	6
E	10	10	15	25	10

(b) Solve the following transportation problem:

	D_1	D_2	D_3	Availability
P_1	300	360	425	600
P_2	390	340	310	300
P_3	255	295	275	1000
Requirement	400	500	800	

(c) Solve the following game graphically:

$$\text{Player } A \quad \overset{\text{Player } B}{\begin{bmatrix} 1 & 2 \\ 5 & 4 \\ -7 & 9 \\ -4 & -3 \\ 2 & 1 \end{bmatrix}}$$

5. (a) Consider the game with the following payoff matrix:

$$\text{Player } A \quad \begin{matrix} A_1 \\ A_2 \end{matrix} \overset{\text{Player } B}{\overset{B_1 \quad B_2}{\begin{bmatrix} 2 & 6 \\ -2 & \lambda \end{bmatrix}}}$$

(i) Show that the game is strictly determinable, whatever λ may be.

(ii) Determine the value of the game.

(b) Solve the following game:

	I	II	III	IV
I	3	2	4	0
II	3	4	2	4
III	4	2	4	0
IV	0	4	0	8

(c) Convert the following game, involving two-person, zero-sum game, into a linear programming problem and hence solve it.

$$\text{Player } A \quad \overset{\text{Player } B}{\begin{bmatrix} 5 & 7 & 2 \\ 10 & 4 & 9 \\ 6 & 2 & 0 \end{bmatrix}}$$

This question paper contains 8 printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : 6626

Unique Paper Code : 32351501 HC

Name of the Paper : C11-Metric Spaces

Name of the Course : B.Sc. (Hons.) Mathematics

Semester : V

Duration : 3 Hours Maximum Marks : 75

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *two* parts from each question.

All questions are compulsory.

1. (*a*) Let $d_p (p \geq 1)$ on the set $\mathbf{R}^n$, be given by

$$d_p(x, y) = \left(\sum_{j=1}^{n} |x_j - y_j|^p \right)^{1/p},$$

for all $x = (x_1, x_2, \ldots\ldots, x_n)$, $y = (y_1, y_2, \ldots\ldots, y_n)$ in $\mathbf{R}^n$.

Show that $(\mathbf{R}^n, d_p)$ is a metric space. Does d_p define a metric on $\mathbf{R}^n$, when $0 < p < 1$? 4+2=6

P.T.O.

(b) Let S be any non-empty set and B(S) denote the set of all real- or complex-valued functions on S, each of which is bounded. Define the uniform metric d on B(S). Show that (B(S), d) is a complete metric space. 6

(c) (i) Let X = **N**, the set of natural numbers. Define

$$d(m, n) = \left|\frac{1}{m} - \frac{1}{n}\right|, \; m, n \in X.$$

Show that (X, d) is an incomplete metric space.

(ii) Prove that metric spaces, **R** with the usual metric and $(0, \infty)$ with the usual metric induced from **R** are homeomorphic. 4+2=6

2. (a) (i) Let (X, d) be a metric space. Prove that the closed ball $\overline{S}(x, r)$, where $x \in X$ and $r > 0$, is a closed subset of X.

(ii) Is the set $A = \{(x, y) : x + y = 1\}$ open in the metric space (R^2, d_2) ? Justify your answer. 4+2=6

(b) Let (X, d) be a metric space and Y a subspace of X. Let Z be a subset of Y. Prove that Z is closed in Y if and only if there exists a closed set F of X such that $X = F \cap Y$. 6

(c) (i) Let (X, d) be a metric space and F_1 and F_2 be subsets of X. Prove that :

$$cl(F_1 \cup F_2) = cl(F_1) \cup cl(F_2).$$

(ii) Define a separable metric space. Is the discrete metric space (X, d) separable ? Justify your answer. 3+3=6

3. (a) Let (X, d) be a metric space and for every nested sequence $\{F_n\}$, $n \geq 1$ of non-empty closed subsets of X such that $d(F_n) \to 0$ as $n \to \infty$, the intersection $\bigcap_{n=1}^{\infty} F_n$

contains one and only one point. Prove that (X, d) is a complete metric space. Further, show that the condition : $d(F_n) \to 0$ as $n \to \infty$ in the above statement can't be dropped. 4+2=6

(b) Let (X, d) be a metric space and $F \subseteq X$. Prove that a point x_0 is a limit point of F if and only if it is possible to select from the set F a sequence, $\{x_n\}$, $n \geq 1$, of distinct points such that $\lim_n d(x_n, x_0) = 0$. 6

(c) (i) Let F be subset of the metric space (X, d). Prove that the set of limit points of F is a closed subset of (X, d).

(ii) Let F be a non-empty bounded closed subset of **R**, with usual metric and $a = \sup F$. Show that $a \in F$. 3+3=6

4. (a) (i) Let (X, d) be any metric space and $f : (X, d) \to (R^n, d_2)$, be defined by :

$$f(x) = (f_1(x), f_2(x), \ldots\ldots, f_n(x)), \text{ for } x \in X.$$

Show that if f is continuous, so is each $f_k : X \to \mathbf{R}$, $k = 1, 2, \ldots\ldots, n$.

(ii) Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be continuous on X. Show that $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$, for all subsets B of Y.

2½+4=6½

(b) Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be uniformly continuous. Show that if $\{x_n\}$, $n \geq 1$, is a Cauchy sequence in X, then so is $\{f(x_n)\}$, $n \geq 1$, in Y. Is this result true, if $f : X \to Y$ is continuous on X ? 4+2½=6½

(c) Let X be the set of all continuous functions defined on [0, 1]. For $f, g \in X$, define the metrics 'd' and 'e' on X by :

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\}, \text{ and}$$

$$e(f, g) = \int_0^1 |f(x) - g(x)|\, dx.$$

Show that these metrics are not equivalent. 6½

5. (a) (i) Let (X, d) be the complete metric space and $T : X \to X$ be a contraction mapping and let $x_0 \in X$ and $\{x_n\}$, $n \geq 1$, be the sequence defined iteratively by $x_{n+1} = T x_n$ for $n = 0, 1, 2, \ldots$. Show that the sequence $\{x_n\}$, $n \geq 1$, is convergent in X.

(ii) Let $T : X \to X$, where (X, d) is a complete metric space, satisfy the inequality :

$$d(Tx, Ty) < d(x, y) \text{ for all } x, y \in X.$$

Show that T need not have a fixed point.

4+2½=6½

(b) Let (**R**, d) be the space of real numbers with the usual metric. Show that a subset, I, of **R** is connected if and only if I is an interval. 6½

(c) (i) Show that the metric space (X, d) is disconnected if and only if there exists a proper subset of X that is both open and closed in X.

(ii) Let A be a subset of $\mathbf{R}^2$ defined by

$$A = \{(x, y) : x^2 - y^2 \geq 4\}.$$

Show that A is disconnected. 4+2½=6½

6. (*a*) Let (X, d_X) be a metric space and every continuous function $f : (X, d_X) \rightarrow (\mathbf{R}, d_u)$, where d_u is the usual metric of $\mathbf{R}$, has the intermediate value property. Prove that (X, d_X) is a connected space. 6½

(*b*) Define the finite intersection property. Prove that a metric space (X, d) is compact if, and only if every collection of closed sets in (X, d) with the finite intersection property has non-empty intersection. 6½

(*c*) Let f be a continuous function from a *compact* metric space (X, d_X) into a metric space (Y, d_Y). Prove that the range $f(X)$ is also compact. 6½

This question paper contains 4 printed pages.

Your Roll No. ....

Sl. No. of Ques. Paper: **6627** **HC**

Unique Paper Code **: 32351502**

Name of Paper **: Group Theory – II**

Name of Course **: B.Sc. (Hons.) Mathematics**

Semester **: V**

Duration **: 3 hours**

Maximum Marks **: 75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

All questions are compulsory. Question No. 1 has been divided into 10 parts and each part is of 1.5 marks. Each question from Q. Nos. 2 to 6 has 3 parts and each part is of 6 marks. Attempt any two parts from each question.

(1) State true (T) or false (F). Justify your answer in brief.

(a) $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is isomorphic to $\mathbb{Z}_4$ here $\mathbb{Z}_n$ is used for group $\{0, 1, 2, \ldots, n-1\}$under the addition modulo n..

(b) The dihedral group D_8 of order 8 is not isomorphic to the quaternion group Q_8 of order 8.

(c) The external direct product $G \oplus H$ is cyclic if and only if groups G and H are cyclic.

(d) $U(165)$ can be written as an external direct product of cyclic additive groups of the form $\mathbb{Z}_n$ where $U(n)$ denotes the group of units under multiplication modulo n.

P. T. O.

(e) Translations $z \mapsto z + a$ are the only automorphisms of the additive group of integers $\mathbb{Z}$.

(f) A subgroup N of a group G is called a characteristic subgroup if $\phi(N) = N$ for all isomorphism of G onto itself.

(g) The number of isomorphism types (classes) of a group of order 9 is 3.

(h) If G is a finite group of order n, then G is isomorphic to a subgroup of $D_n n$.

(i) If a group G acts trivially on a set A containing more than 1 elements then there is an element a in A whose stabilizer is proper subgroup of the group.

(j) $U(8)$ is isomorphic to $U(10)$.

(2) (a) Find all subgroups of order 3 in $\mathbb{Z}_9 \oplus \mathbb{Z}_3$.

(b) Prove that for any group G , $G/Z(G)$ is isomorphic to the group of inner automorphism $Inn(G)$ where $Z(G)$ is centre of the group G.

(c) Classify groups of order 6.

(3) (a) (i) Suppose that G is a group of order 4 with identity e and $x^2 = e$ for all x in G. Prove that G is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

(ii) Find two groups G_1 and G_2 such that G_1 is isomorphic to G_2 but $Aut(G_1)$ is not isomorphic to $Aut(G_2)$ where $Aut(G_i)$ is the group of automorphisms of G_i.

(b) (i) Suppose that N is a normal subgroup of a finite group G. If G/N has an element of order n , show that G has an element of order n. Also show, by an example, that the assumption that G is finite is necessary.

(ii) If G is a non abelian group then show that $Aut(G)$ is not cyclic.

(c) Define the characteristic subgroup of a group G. Prove that every subgroup of a cyclic group is characteristic.

(4) (a) If p is a prime and G is a group of prime power order p^α for some positive integer $\alpha \geq 1$, then show that G has a non trivial centre.

(b) Find all conjugacy classes of the dihedral group D_8 of order 8 and the quaternion group Q_8 of order 8 and hence verify the class equation.

(c) Prove that if H has a finite index n in G then there is a normal subgroup K of G where K is subgroup of H and the index of K in G ($|G : K|$) is less than or equal to $n!$.

(5) (a) Prove that if p is a prime and G is a group of order p^α for some positive integer α then every subgroup of index p is normal in G. Deduce that every group of order p^2 has a normal subgroup of order p.

(b) Prove that a group of order 56 has a normal Sylow p-subgroup for some prime p dividing its order.

(c) Prove that two elements of the symmetric group on n letters S_n are conjugate in S_n if and only if they have same cycle type.Also show that the number of conjugacy classes equals the number of partitions of n.

(6) (a) Define a simple group. Prove that if G is an abelian simple group then G is isomorphic to the cyclic group $\mathbb{Z}_p$ for some prime p.

(b) (i) Prove that group of order 280 is not simple.

(ii) Show that the alternating group A_5 of degree 5 can not contain a subgroup of order 30 or 20 or 15.

(c) (i) If the centre of G is of index n, then prove that every conjugacy class has at most n elements.

(ii) Prove that the centre $Z(S_n)$ of symmetric group S_n contains only the identity of S_n for all n greater than or equal to three.